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Lax, Korteweg-de Vries, Schrödinger, (sinh-)Gordon:
Integrable systems and spectral theory

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Abstract. The topic of this pair of lectures is the spectral theory for integrable systems. It is not easy to give a precise definition of when one calls a dynamical system integrable; in the words of Hitchin[1], "integrability of a system of differential equations should manifest itself through some generally recognizable features: the existence of many conserved quantities; the presence of algebraic geometry; the ability to give explicit solutions." For the present lectures we are interested in infinite-dimensional integrable systems defined by a partial differential equation in two variables. In the context of integrable systems, "spectral theory" refers to an assortment of methods deriving conserved quantities for integrable systems by means of eigenvalue problems or their generalisations; through such methods it is often possible to design algebraic data called "spectral data" which characterize an individual periodic solution of the underlying differential equation uniquely. In an infinite-dimensional integrable system, we expect infinitely many spectral quantities, and questions of convergence and asymptotic behavior plays a major role in their investigation.

It is a remarkable feature in the study of integrable systems that there is very little general theory that applies to all integrable systems. This is partially due to the vague nature of the concept of an integrable system. For example, with respect to spectral theory there is no unified definition of the associated objects that "works" in all integrable systems. Rather, their definition needs to be adapted to the individual integrable system under investigation. Although the results one obtains in this way for the various integrable systems are very similar to one another, because of these differences there does not exist a unified proof of these results. Rather, they need to be proven for each integrable system separately.

For this reason we will discuss three individual integrable systems and the associated spectral theory.

1. The **Korteweg-de Vries (KdV)** equation is the partial differential equation for a function $u = u(x,t)$ in two real variables
$$u_t + u_{xxx} - 6\,u\,u_x = 0\ .$$
This equation is named for the two Dutch mathematicians Korteweg and de Vries, who studied it as a mathematical model of waves on a shallow water surface. The associated integrable system is the one which sparked the interest in integrable systems, for which most of the aspects of integrable systems theory were first discovered, and which probably remains the most studied infinite-dimensional integrable system today. In particular, in 1968, P. Lax made the fundamental discovery that the KdV equation can be written in what is today called Lax form, i.e. in the form
$$\frac{\mathrm{d}}{\mathrm{d}t}L = [B, L]\ , \qquad (*)$$
where L and B are two differential operators depending on the solution u. For the KdV equation, L is the 1-dimensional Schrödinger operator
$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u\ ,$$

[1] H. J. Hitchin et al., *Integrable Systems*, Oxford 1999, p. 1

and

$$B = 4\frac{\mathrm{d}^3}{\mathrm{d}x^3} - 3u\frac{\mathrm{d}}{\mathrm{d}x} - 3\frac{\mathrm{d}}{\mathrm{d}x}u \ .$$

It is known that the periodic spectrum of the operator $L = L(t)$ is a pure point spectrum, and therefore equal to the set of periodic eigenvalues. We will see that the spectrum of $L(t)$ is conserved in the sense that it does not depend on t. We will use it to construct "spectral data" for periodic solutions of the KdV equation.

The second order differential operator L can be rewritten as a 2-dimensional first order differential operator, and then the eigenvalue equation $L\psi = \lambda\psi$ takes the form $\frac{d}{dx}\Psi = U_\lambda \Psi$ with a 2-dimensional periodic operator U_λ depending on the eigenvalue λ. There moreover exists another 2-dimensional periodic operator V_λ (also depending on λ) so that the Lax equation $(*)$ is equivalent to the zero curvature condition

$$\frac{\partial U_\lambda}{\partial t} - \frac{\partial V_\lambda}{\partial x} + [U_\lambda, V_\lambda] = 0 \ . \qquad (\dagger)$$

2. The 2-dimensional **(self-focusing) non-linear Schrödinger equation (sfNLS)** is the partial differential equation

$$u_{xx} - 2i\, u_t + 2|u|^2\, u = 0 \ .$$

Its original significance comes from the physics of quantum mechanics, but it also plays a role in the reconstruction of a curve in a 3-dimensional space form from its (Hasimoto) complex curvature. This differential equation can also be written in Lax form, which gives rise to a spectral theory in a similar way as with the KdV equation.

3. The 2-dimensional **sinh-Gordon equation** is the partial differential equation

$$\triangle u + \sinh(u) = 0 \ .$$

An immersed surface in a 3-dimensional space form without umbilic points has constant mean curvature if and only if near every point of the surface, there exists a conformal coordinate z so that the conformal factor u of the induced metric $e^{u/2}\,\mathrm{d}z\,\mathrm{d}\overline{z}$ is a solution of the sinh-Gordon equation. This partial differential equation can not be written in Lax form. However, the zero curvature condition $(\dagger)$ does generalize to the integrable system of the sinh-Gordon equation. Although it can no longer be interpreted as an eigenvalue equation for some differential operator (because the dependency of the operators U_λ, V_λ on λ is more complicated), we will see that it gives rise to a theory that is analogous to the spectral theory of the preceding two cases, and we will take the liberty of applying the label of "spectral theory" also here.